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EVOLUTION OF THE JOINT MOTION OF TWO VISCOUS HEAT-CONDUCTING FLUIDS IN A PLANE LAYER UNDER THE ACTION OF AN UNSTEADY PRESSURE GRADIENT

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A study is made of an invariant solution of the equations of a viscous heat-conducting fluid, which is treated as unidirectional motion of two such fluids in a plane layer with a common boundary under the action of an unsteady pressure gradient. A priori estimates of the velocity and temperature are obtained. The steady state is determined, and it is shown (under some conditions on the pressure gradient) that, at larger times, this state is the limiting one. For semiinfinite layers, a solution in closed form is obtained using the Laplace transform.

Key words: *viscous heat-conducting fluid, interface, steady-state flow.*

1. Formulation of the Problem. Motion of two immiscible incompressible viscous heat-conducting fluids with a common interface is considered. We introduce the following notation: Ω_j $(j = 1, 2)$ are the domains occupied by the fluids with interface Γ, $\boldsymbol{u}_j(\boldsymbol{x}, t)$ and $p_j(\boldsymbol{x}, t)$ are the velocity vector and pressure, respectively, and $\theta_j(\boldsymbol{x}, t)$ are deviations from average temperature. Then, in the absence of external forces, the system of equations is written as

$$\frac{d\boldsymbol{u}_j}{dt} + \frac{1}{\rho_j}\nabla p_j = \nu_j \Delta \boldsymbol{u}_j, \qquad \frac{d\theta_j}{dt} = \chi_j \Delta\theta_j, \qquad \operatorname{div} \boldsymbol{u}_j = 0, \tag{1.1}$$

where ρ_j is the average density, ν_j is the kinematic viscosity, χ_j is the thermal diffusivity, and $d/dt = \partial/\partial t + \boldsymbol{u}_j \cdot \nabla$.

At the interface Γ, we specify the following conditions:

— the equality of the velocities

$$\boldsymbol{u}_1 = \boldsymbol{u}_2, \qquad \boldsymbol{x} \in \Gamma; \tag{1.2}$$

— the kinematic condition

$$\boldsymbol{u}_j \cdot \boldsymbol{n} = V_{\boldsymbol{n}}, \qquad \boldsymbol{x} \in \Gamma; \tag{1.3}$$

— the dynamic condition (in the case of no surface tension)

$$(P_2 - P_1)\boldsymbol{n} = 0, \qquad \boldsymbol{x} \in \Gamma; \tag{1.4}$$

— the continuity condition for the temperature and heat flux

$$\theta_1 = \theta_2, \qquad k_2 \frac{\partial\theta_2}{\partial n} - k_1 \frac{\partial\theta_2}{\partial n} = 0, \qquad \boldsymbol{x} \in \Gamma. \tag{1.5}$$

In (1.2)–(1.5), $\boldsymbol{n}$ is the unit normal vector to the surface Γ directed from the domain Ω_1 to the domain Ω_2, $V_{\boldsymbol{n}}$ is the velocity of motion of the surface along the normal, $P_j = -p_j E + 2\rho_j\nu_j D(\boldsymbol{u}_j)$ are the stress tensors, D is the strain rate tensor, and the constant k_j is the thermal conductivity.

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The domains Ω_1 and Ω_2 can be in contact not only with each other but also with the solid walls Σ_j. On the walls, we impose the slip condition

$$\boldsymbol{u}_j = \boldsymbol{a}_j(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \Sigma_j, \tag{1.6}$$

where $\boldsymbol{a}_j(\boldsymbol{x}, t)$ is the velocity of motion of the wall Σ_j. In addition, the temperature on Σ_j is considered specified:

$$\theta_j = \theta_w^j(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \Sigma_j. \tag{1.7}$$

To complete the formulation of the problem, relations (1.1)–(1.7) need to be supplemented by the initial conditions

$$\boldsymbol{u}_j(\boldsymbol{x}, 0) = \boldsymbol{u}_{0j}(\boldsymbol{x}), \qquad \operatorname{div} \boldsymbol{u}_{0j} = 0, \qquad \theta_j(\boldsymbol{x}, 0) = \theta_{0j}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega_j.$$

Below, we consider the system of equations governing the two-dimensional motion of two fluids with a plane interface. It can be shown that this system admits the one-parameter subgroup [1] corresponding to the operator

$$\frac{\partial}{\partial x} + A_j \frac{\partial}{\partial \theta_j} - \rho_j f_j(t) \frac{\partial}{\partial p_j}$$

[A_j are constants and $f_j(t)$ are functions of time]. The invariant solution should be sought in the form

$$u_j = u_j(y, t), \quad v_j = v_j(y, t), \quad p_j = -\rho_j f_j(t) x + P_j(y, t), \quad \theta_j = A_j x + T_j(y, t).$$

The equation of conservation of mass implies that v_j depends only on time: $v_j = v_j(t)$, and the projection of the momentum equations onto the y axis implies the relation $\rho_j^{-1} P_{jy} = v_{jt}(t)$, where the subscripts y and t denote partial derivatives with respect to the corresponding variables. We will assume that $v_j(t) = 0$; otherwise the slip conditions on the motionless walls are not satisfied. Thus, the invariant solution is represented as

$$u_j = u_j(y, t), \quad v_j = 0, \quad p_j = \rho_j f_j(t) x + P_j(t), \quad \theta_j = A_j x + T_j(y, t). \tag{1.8}$$

Substituting (1.8) into system (1.1) and taking into account conditions (1.2)–(1.5) on the interface $y = 0$, we obtain the initial-boundary-value problem

$$\begin{gathered} u_{jt} = \nu_j u_{jyy} + f_j(t), \qquad T_{jt} = \chi_j T_{jyy} - A u_j \\ \text{at} \quad -l_1 < y < 0 \ (j = 1), \quad 0 < y < l_2 \ (j = 2); \end{gathered} \tag{1.9}$$

$$\begin{gathered} u_1(0, t) = u_2(0, t), \qquad T_1(0, t) = T_2(0, t), \\ k_1 T_{1y}(0, t) = k_2 T_{2y}(0, t), \qquad \rho_2 \nu_2 u_{2y}(0, t) - \rho_1 \nu_1 u_{1y}(0, t) = 0, \\ u_j(y, 0) = 0, \qquad T_j(y, 0) = 0. \end{gathered} \tag{1.10}$$

In the second equation of (1.9), $A \equiv A_1 = A_2$ (due to the equality of the temperatures at $y = 0$). Conditions (1.10) need to be supplemented by conditions on the solid walls $y = -l_1$ and $y = l_2$ (1.6):

$$u_1(-l_1, t) = u_2(l_2, t) = 0 \tag{1.11}$$

and zero temperature perturbations (1.7) should be specified:

$$T_1(-l_1, t) = 0, \qquad T_2(l_2, t) = 0. \tag{1.12}$$

Thus, solution (1.8) can be interpreted as follows. We assume that, at the initial time, the first fluid fills a layer $-l_1 < y < 0$ and the second a layer $0 < y < l_2$. The fluids are in the state of rest in each layer of the temperature field $\theta_j = Ax$. Instantaneously arising pressure gradients $f_j(t)$ sets the fluids in motion in which the interface remains plane ($y = 0$) and the trajectories are straight lines parallel to the x axis. The functions u_j and T_j will be called perturbations of the state of rest of the fluids. At $A \neq 0$, the velocity field influences the temperature perturbation in the layers $(-l_1, 0)$ and $(0, l_2)$. The evolution of such perturbations is described by the solution of the initial-boundary-value problem (1.9)–(1.12).

It should be noted that the unidirectional (layered) motion of a viscous fluid under the action of a pressure gradient have been studied extensively (see, for example, [2, 3]). The velocity field, as a rule, is represented in the form of a number of channels of finite width. In the case of motion of two viscous fluids with a common interface for semibounded layers, self-similar solutions describing the smoothing of a plane velocity discontinuity [3] and thermocapillary motion [1] have been found.

REMARK 1. Because $p_1 = p_2$ at $y = 0$ for all x, the dynamic condition at the interface (1.4) implies that

$$\rho_1 f_1(t) = \rho_2 f_2(t), \qquad P_1(t) = P_2(t). \tag{1.13}$$

Thus, Eqs. (1.9)–(1.12) form two successively solved problems for the functions (u_1, u_2) and (T_1, T_2).

2. Determination of the Velocity Field in the Layers. Taking into account Remark 1, we examine the problem of the velocity field in one of the layers for a pressure gradient that arises suddenly in one of the layers. In this case, we have the linear conjugate initial-boundary-value problem [$f(t) \equiv f_1(t)$]:

$$u_{1t} = \nu_1 u_{1yy} + f(t), \qquad -l_1 < y < 0; \tag{2.1}$$

$$u_1(-l_1, t) = 0; \tag{2.2}$$

$$u_{2t} = \nu_2 u_{2yy} + (\rho_1/\rho_2) f(t), \qquad 0 < y < l_2; \tag{2.3}$$

$$u_2(l_2, t) = 0; \tag{2.4}$$

$$u_1(0, t) = u_2(0, t), \qquad \mu_1 u_{1y}(0, t) = \mu_2 u_{2y}(0, t), \qquad t \geq 0; \tag{2.5}$$

$$u_1(y, 0) = 0, \quad -l_1 < y < 0, \qquad u_2(y, 0) = 0, \quad 0 < y < l_2. \tag{2.6}$$

Here $\mu_{1,2} = \rho_{1,2}\nu_{1,2}$ are the dynamic viscosities.

REMARK 2. Without loss of generality in (1.13), we can assume that $P_1(t) = P_2(t) = 0$ since these functions do not influence the motion of the fluids.

Let us obtain some a priori estimates of the solution of problem (2.1)–(2.6). We multiply Eq. (2.1) by $\rho_1 u_1(y, t)$ [Eq. (2.3) by $\rho_2 u_2(y, t)$] and integrate the result over y from $-l_1$ to zero (from zero to l_2). Combining the resulting equalities and using boundary conditions (2.2), (2.4), and (2.5), we obtain the relation

$$\frac{dE(t)}{dt} + \mu_1 \int\limits_{-l_1}^{0} u_{1y}^2\, dy + \mu_2 \int\limits_{0}^{l_2} u_{2y}^2\, dy = \rho_1 f(t) \Big(\int\limits_{-l_1}^{0} u_1\, dy + \int\limits_{0}^{l_2} u_2\, dy \Big), \tag{2.7}$$

where

$$E(t) = \frac{1}{2}\,\rho_1 \int\limits_{-l_1}^{0} u_1^2(y, t)\, dy + \frac{1}{2}\,\rho_2 \int\limits_{0}^{l_2} u_2^2(y, t)\, dy \tag{2.8}$$

is the kinetic energy of the two layers.

Equation (2.7), in particular, implies that the solution of problem (2.1)–(2.6) is unique: if $f(t) = 0$, then $u_1(y, t) = u_2(y, t) \equiv 0$.

Equality (2.7) allows one to determine [with some constraints on the function $f(t)$] the asymptotic behavior of the solution as $t \to \infty$. Indeed, by virtue of conditions (2.2) and (2.4) for $u_1(y, t)$ and $u_2(y, t)$, Friedrichs inequalities are valid:

$$\int\limits_{-l_1}^{0} u_1^2(y, t)\, dy \leq \frac{l_1^2}{2} \int\limits_{-l_1}^{0} u_{1y}^2(y, t)\, dy, \qquad \int\limits_{0}^{l_2} u_2^2(y, t)\, dy \leq \frac{l_2^2}{2} \int\limits_{0}^{l_2} u_{2y}^2(y, t)\, dy. \tag{2.9}$$

Using inequalities (2.9) and the Cauchy–Bunyakovskii inequality and taking into account that $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$, $a \geq 0$, $b \geq 0$, from (2.7) we obtain

$$\frac{dE(t)}{dt} + 4\delta E(t) \leq 2\delta_1 |f(t)| \sqrt{E(t)}\,, \tag{2.10}$$

where $\delta = \min\,(l_1^{-2}\nu_1, l_2^{-2}\nu_2)$ and $\delta_1 = \rho_1 \max\,(\sqrt{l_1/\rho_1}, \sqrt{l_2/\rho_2})$. According to (2.8) and initial conditions (2.6), $E(0) = 0$, and, hence, from (2.10) we obtain

$$E(t) \le \delta_1^2 \Big(\int\limits_0^t |f(t)|\, \mathrm{e}^{2\delta t}\, dt \Big)^2 \mathrm{e}^{-4\delta t}\,. \tag{2.11}$$

Hence, if the integral

$$\int\limits_0^\infty |f(t)|\, \mathrm{e}^{2\delta t}\, dt \equiv \sqrt{C_1} > 0 \tag{2.12}$$

converges, then, Eq. (2.11) implies the inequality

$$E(t) \le \delta_1^2 C_1\, \mathrm{e}^{-4\delta t} \tag{2.13}$$

for all $t \ge 0$. Therefore, as $t \to \infty$, the L^2-norms of the functions $u_1(y,t)$ and $u_2(y,t)$ tend to zero exponentially and uniformly in $y \in (-l_2, 0)$ and $y \in (0, l_2)$ if condition (2.12) is satisfied. To obtain estimates of $|u_j(y,t)|$, it is necessary to estimate the integrals

$$\int\limits_{-l_1}^0 u_{1y}^2\, dy, \qquad \int\limits_0^{l_2} u_{2y}^2\, dy.$$

Let $u(y,t)$ be a solution of the equation $u_t = \nu u_{yy} + F(y,t)$, $y \in [a,b]$. Then, the following identity holds:

$$\int\limits_0^t \int\limits_a^b (u_t^2 + \nu^2 u_{yy}^2)\, dy\, dt + \nu \int\limits_a^b u_y^2\, dy$$

$$= 2\nu \int\limits_0^t (u_t u_y)\Big|_a^b dt + \nu \int\limits_a^b u_{0y}^2\, dy + \int\limits_0^t \int\limits_a^b F^2(y,t)\, dy\, dt, \tag{2.14}$$

where $u_0(y) = u(y,0)$. Identity (2.14) follows from the equality

$$\int\limits_0^t \int\limits_a^b (u_t - \nu u_{yy})^2\, dy\, dt = \int\limits_0^t \int\limits_a^b F^2(y,t)\, dy\, dt, \qquad u_t u_{yy} = \frac{\partial}{\partial y}\,(u_t u_y) - \frac{1}{2}\,\frac{\partial}{\partial t}\,(u_y^2).$$

In (2.14) we first set $u = u_1$, $a = -l_1$, $b = 0$, $\nu = \nu_1$, and $F = f(t)$ and multiply the resulting equality by ρ_1; after that, we set $u = u_2$, $a = 0$, $b = l_2$, $\nu = \nu_2$, and $F = \rho_1 \rho_2^{-1} f(t)$ and multiply the resulting equality by ρ_2. Combining these equalities, for problem (2.1)–(2.6) we obtain the integral identity

$$\rho_1 \int\limits_0^t \int\limits_{-l_1}^0 (u_{1t}^2 + \nu_1^2 u_{1yy}^2)\, dy\, dt + \rho_2 \int\limits_0^t \int\limits_0^{l_2} (u_{2t}^2 + \nu_2^2 u_{2yy}^2)\, dy\, dt$$

$$+ \mu_1 \int\limits_{-l_1}^0 u_{1y}^2\, dy + \mu_2 \int\limits_0^{l_2} u_{2y}^2\, dy = \rho_1 (l_1 + l_2) \int\limits_0^t f^2(t)\, dt. \tag{2.15}$$

Equation (2.15) was derived taking into account boundary conditions (2.2), (2.4), and (2.5) and initial conditions (2.6). Hence, for all $t \ge 0$, the inequalities

$$\int\limits_{-l_1}^0 u_{1y}^2\, dy \le \frac{E_1(t)}{\mu_1}, \qquad \int\limits_0^{l_2} u_{2y}^2\, dy \le \frac{E_1(t)}{\mu_2} \tag{2.16}$$

are valid [$E_1(t)$ is the right side of (2.15)]. Then, if in addition to the integral in (2.12), the integral

$$\int\limits_0^\infty f^2(t)\, dt \equiv C_2 > 0 \tag{2.17}$$

converges, the following estimates hold, which are uniform in y [$y \in (-l_1, 0)$ and $y \in (0, l_2]$:

$$|u_j(y,t)| \le \Big(2\delta_1 \sqrt{\frac{2C_1C_3}{\mu_j\rho_j}}\Big)^{1/2} \mathrm{e}^{-2\delta t} \tag{2.18}$$

[$C_3 = \rho_1(l_1 + l_2)C_2$; $j = 1, 2$]. Estimates (2.18) are obtained using the equalities

$$u_1^2(y,t) = 2\int\limits_{-l_1}^{y} u_1(y,t)u_{1y}(y,t)\,dy, \qquad u_2^2(y,t) = -2\int\limits_{y}^{l_2} u_2(y,t)u_{2y}(y,t)\,dy,$$

the inequalities (2.7), (2.16), and (2.17), and the Cauchy–Bunyakovskii inequalities.

REMARK 3. It can be shown that if condition (2.12) is satisfied, condition (2.17) is also satisfied.

Thus, we proved the following theorem.

Theorem 1. *If the condition* (2.12) *is satisfied and* $t \to \infty$, *the solution of problem* (2.1)–(2.6) *tends to the zero solution, and the estimates of the rate of convergence* (2.18), *which are uniform in the intervals* $(-l_1, 0)$ *and* $(0, l_2)$, *are valid.*

In other words, if in one of the fluids, the pressure gradient tends to zero rapidly enough, then, according to inequalities (2.18), the motion of these fluids is retarded by viscous friction.

To gain more accurate information on the behavior of $u_j(y,t)$, we apply the Laplace transform to problem (2.1)–(2.6):

$$\tilde{u}_j(y,p) = \int\limits_0^{\infty} \mathrm{e}^{-pt}\, u_j(y,t)\,dt, \qquad j = 1, 2; \tag{2.19}$$

for the conditions of applicability of formula (2.19) see, for example, in [4]. As a result, we obtain the following boundary-value problem for the images $\tilde{u}_j(y,p)$:

$$\begin{gathered} \tilde{u}_1'' - \frac{p}{\nu_1}\,\tilde{u}_1 = -\frac{\tilde{f}(p)}{\nu_1} \quad (-l_1 < y < 0), \qquad \tilde{u}_1(-l_1, p) = 0, \\ \tilde{u}_2'' - \frac{p}{\nu_2}\,\tilde{u}_2 = -\frac{\rho_1}{\rho_2\nu_2}\,\tilde{f}(p) \quad (0 < y < l_2), \qquad \tilde{u}_2(l_2, p) = 0, \\ \tilde{u}_1(0,p) = \tilde{u}_2(0,p), \qquad \mu_1\tilde{u}_1'(0,p) = \mu_2\tilde{u}_2'(0,p) \end{gathered} \tag{2.20}$$

(the prime denotes differentiation with respect to y).

After some transformations, from (2.20), we obtain

$$\begin{gathered} \tilde{u}_1(y,p) = -\frac{\tilde{f}(p)}{pW(p)}\Big[\Big(\rho - (\rho-1)\cosh\sqrt{\frac{p}{\nu_2}}\,l_2\Big)\sinh\sqrt{\frac{p}{\nu_1}}\,(y + l_1) \\ - \Big(\sinh\sqrt{\frac{p}{\nu_1}}\,y + \sinh\sqrt{\frac{p}{\nu_1}}\,l_1\Big)\cosh\sqrt{\frac{p}{\nu_2}}\,l_2 + \frac{\mu}{\sqrt{\nu}}\Big(\cosh\sqrt{\frac{p}{\nu_1}}\,y - \cosh\sqrt{\frac{p}{\nu_1}}\,l_1\Big)\sinh\sqrt{\frac{p}{\nu_2}}\,l_2\Big], \\ \tilde{u}_2(y,p) = -\frac{\tilde{f}(p)}{pW(p)}\Big[\frac{\mu}{\sqrt{\nu}}\Big(1 + (\rho-1)\cosh\sqrt{\frac{p}{\nu_1}}\,l_1\Big)\sinh\sqrt{\frac{p}{\nu_2}}\,(l_2 - y) \\ + \frac{\mu}{\sqrt{\nu}}\,\rho\Big(\sinh\sqrt{\frac{p}{\nu_2}}\,y - \sinh\sqrt{\frac{p}{\nu_2}}\,l_2\Big)\cosh\sqrt{\frac{p}{\nu_1}}\,l_1 + \rho\Big(\cosh\sqrt{\frac{p}{\nu_2}}\,y - \cosh\sqrt{\frac{p}{\nu_2}}\,l_2\Big)\sinh\sqrt{\frac{p}{\nu_1}}\,l_1\Big]. \end{gathered} \tag{2.21}$$

Here $\tilde{f}(p)$ is the image of $f(t)$, $\rho = \rho_1/\rho_2$, $\mu = \mu_1/\mu_2$, $\nu = \nu_1/\nu_2$, and

$$W(p) = \sinh\sqrt{\frac{p}{\nu_2}}\,l_2\cosh\sqrt{\frac{p}{\nu_1}}\,l_1\Big(\frac{\mu}{\sqrt{\nu}} + \coth\sqrt{\frac{p}{\nu_2}}\,l_2\tanh\sqrt{\frac{p}{\nu_1}}\,l_1\Big). \tag{2.22}$$

The originals $u_j(y,t)$ $(j=1,2)$ are found from the formula

$$u_j(y,t)=\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\mathrm{e}^{pt}\,\tilde u_j(y,p)\,dp. \tag{2.23}$$

We assume that there exists the limit $\lim\limits_{t\to\infty}f(t)=f_0=\text{const}$. Then, $\lim\limits_{p\to 0}p\tilde f(p)=f_0$ [4]. It is clear that, in this case, the function $f(t)$ does not satisfy condition (2.12). We calculate $\lim\limits_{p\to 0}p\tilde u_j(y,p)$ according to formulas (2.21). Simple but tedious calculations taking into account the asymptotic representations $\sinh x\sim x+x^3/6$ and $\cosh x\sim 1+x^2/2$ as $x\to 0$ yield

$$\begin{aligned}\lim_{p\to 0}p\tilde u_1(y,p)&=\frac{l_1^2f_0}{2\nu_1}\Big[-\Big(\frac{y}{l_1}\Big)^2+\frac{\mu-l^2}{l(\mu+l)}\,\frac{y}{l_1}+\frac{\mu(l+1)}{l(\mu+l)}\Big]\equiv u_1^0(y),\\ \lim_{p\to 0}p\tilde u_2(y,p)&=\frac{l_2^2f_0\mu}{2\nu_1}\Big[-\Big(\frac{y}{l_2}\Big)^2+\frac{\mu-l^2}{\mu+l}\,\frac{y}{l_2}+\frac{l(l+1)}{\mu+l}\Big]\equiv u_2^0(y),\end{aligned} \tag{2.24}$$

where $l=l_1/l_2$. It is easy to check that the right sides in (2.24) are the exact steady-state solution of problem (2.1)–(2.6) with the replacement of $f(t)$ by f_0. Thus, as $t\to\infty$, the solution of problem (2.1)–(2.6) become steady-state.

The solution for semibounded layers can be obtained from formulas (2.21). For this, in formulas (2.21) we set $l_1,l_2\to\infty$. Then, according to (2.22)

$$W(p)\sim\Big(1+\frac{\mu}{\sqrt{\nu}}\Big)\exp\Big(\sqrt{\frac{p}{\nu_1}}\,l_1+\sqrt{\frac{p}{\nu_2}}\,l_2\Big).$$

Denoting the limits $\tilde u_j(y,p,l_1,l_2)$ by $\tilde U_j(y,p)$ and performing some transformations, we obtain

$$\begin{aligned}\tilde U_1(y,p)&=\frac{\tilde f(p)}{p}\Big[1+\frac{\sqrt{\nu}\,(\rho-1)}{\mu+\sqrt{\nu}}\exp\Big(\sqrt{\frac{p}{\nu_1}}\,y\Big)\Big],\\ \tilde U_2(y,p)&=\frac{\tilde f(p)}{p}\Big[\rho-\frac{\mu(\rho-1)}{\mu+\sqrt{\nu}}\exp\Big(-\sqrt{\frac{p}{\nu_2}}\,y\Big)\Big].\end{aligned} \tag{2.25}$$

Using the properties of the inverse Laplace transform [4], we find the originals:

$$\begin{aligned}U_1(y,t)&=\int\limits_0^t f(\tau)\Big[1+\frac{\sqrt{\nu}\,(\rho-1)}{\mu+\sqrt{\nu}}\ \mathrm{Erf}\Big(-\frac{y}{2\sqrt{\nu_1(t-\tau)}}\Big)\Big]\,d\tau,\\ U_2(y,t)&=\int\limits_0^t f(\tau)\Big[\rho-\frac{\mu(\rho-1)}{\mu+\sqrt{\nu}}\ \mathrm{Erf}\Big(\frac{y}{2\sqrt{\nu_2(t-\tau)}}\Big)\Big]\,d\tau.\end{aligned} \tag{2.26}$$

Here

$$\mathrm{Erf}\,z=1-\mathrm{erf}\,z,\qquad \mathrm{erf}\,z=\frac{2}{\sqrt{\pi}}\int\limits_0^z\mathrm{e}^{-z^2}\,dz.$$

From formulas (2.26), we obtain the solution of problem (2.1), (2.3), (2.5), (2.6) in semibounded layers.

Instead of the pressure gradient, it is possible to specify the volumetric flow rate in the layers:

$$Q_1(t)=\int\limits_{-l_1}^{0}u_1(y,t)\,dy,\qquad Q_2(t)=\int\limits_0^{l_2}u_2(y,t)\,dy. \tag{2.27}$$

For example, the layer $(-l_1,0)$ is water and the layer $(0,l_2)$ is oil; the oil flow rate $Q_2(t)$ is specified. Applying the Laplace transform (2.19) to equalities (2.27) and using formulas (2.21), we obtain

$$\tilde{Q}_1(p) = -\frac{\tilde{f}(p)}{pW(p)}\Big[\sqrt{\frac{\nu_1}{p}}\Big(\cosh\sqrt{\frac{p}{\nu_1}}\,l_1 - 1\Big)\Big(\rho - (\rho-2)\cosh\sqrt{\frac{p}{\nu_2}}\,l_2\Big)$$

$$+\frac{\mu}{\sqrt{\nu}}\sqrt{\frac{\nu_1}{p}}\sinh\sqrt{\frac{p}{\nu_1}}\,l_1\sinh\sqrt{\frac{p}{\nu_2}}\,l_2$$

$$-l_1\Big(\sinh\sqrt{\frac{p}{\nu_1}}\,l_1\cosh\sqrt{\frac{p}{\nu_2}}\,l_2 + \frac{\mu}{\sqrt{\nu}}\cosh\sqrt{\frac{p}{\nu_1}}\,l_1\sinh\sqrt{\frac{p}{\nu_2}}\,l_2\Big)\Big]; \tag{2.28}$$

$$\tilde{Q}_2(p) = -\frac{\tilde{f}(p)}{pW(p)}\Big[\frac{\mu}{\sqrt{\nu}}\sqrt{\frac{\nu_2}{p}}\Big(\cosh\sqrt{\frac{p}{\nu_2}}\,l_2 - 1\Big)\Big(1 + (2\rho-1)\cosh\sqrt{\frac{p}{\nu_1}}\,l_1\Big)$$

$$+\rho\sqrt{\frac{\nu_2}{p}}\sinh\sqrt{\frac{p}{\nu_2}}\,l_2\sinh\sqrt{\frac{p}{\nu_1}}\,l_1 - \rho l_2\Big(\frac{\mu}{\sqrt{\nu}}\sinh\sqrt{\frac{p}{\nu_2}}\,l_2\cosh\sqrt{\frac{p}{\nu_1}}\,l_1 + \cosh\sqrt{\frac{p}{\nu_2}}\,l_2\sinh\sqrt{\frac{p}{\nu_1}}\,l_1\Big)\Big]. \tag{2.29}$$

From (2.29) we obtain $\tilde{f}(p)$, and from formula (2.23) we find $f(t)$. The flow rate of the first fluid (water) is determined from (2.28) and (2.23).

It is of interest to determine the flow rate for the steady-state flow (2.24). In this case,

$$Q_1^0 = \int\limits_{-l_1}^{0} u_1^0(y)\,dy = \frac{f_0 l_1^3}{12\nu_1 l(\mu+l)}\,(4\mu l + 3\mu + l^2),$$

$$Q_2^0 = \int\limits_{0}^{l_2} u_2^0(y)\,dy = \frac{f_0 l_2^3\mu}{12\nu_1(\mu+l)}\,(\mu + 4l + 3l^2).$$

The ratio

$$\frac{Q_2^0}{Q_1^0} = \frac{\mu}{l^2}\,\frac{\mu+4l+3l^2}{4\mu l+3\mu+l^2}$$

depends greatly on the thickness of the layers. Thus, for water and oil ($\mu = 0.312$), we have $Q_2^0/Q_1^0 \approx 5.71$ at $l = 0.25$ ($l_2 = 4l_1$) and $Q_2^0/Q_1^0 \approx 2.11$ at $l = 0.5$ ($l_2 = 2l_1$).

3. Evolution of Temperature Perturbations. The initial-boundary-value problem has the form

$$T_{1t} = \chi_1 T_{1yy} - Au_1, \qquad -l_1 < y < 0; \tag{3.1}$$

$$T_1(-l_1, t) = 0; \tag{3.2}$$

$$T_{2t} = \chi_2 T_{2yy} - Au_2, \qquad 0 < y < l_2; \tag{3.3}$$

$$T_2(l_2, t) = 0; \tag{3.4}$$

$$T_1(0,t) = T_2(0,t), \qquad k_1 T_{1y}(0,t) = k_2 T_{2y}(0,t); \tag{3.5}$$

$$T_1(y,0) = 0, \qquad T_2(y,0) = 0. \tag{3.6}$$

The formulation of problem (3.1)–(3.6) coincides with the formulation of problem (2.1)–(2.6) in which $f(t)$ needs to be replaced by $-Au_1(y,t)$, $\rho_1\rho_2^{-1}f(t)$ by $-Au_2(y,t)$, ν_j by χ_j, and μ_j by k_j. As $\chi_j = k_j/(\rho_j c_{0j})$ (c_{0j} are the specific heat capacities of the mixtures), multiplying Eq. (3.1) by $\rho_1 c_{01} T_1$ [Eq. (3.3) by $\rho_2 c_{02} T_2$], and integrating over y from $-l_1$ to 0 (from 0 to l_2) and combining the equalities obtained, similarly to (2.7) we find

$$\frac{dE_2}{dt} + k_1\int\limits_{-l_1}^{0} T_{1y}^2\,dy + k_2\int\limits_{0}^{l_2} T_{2y}^2\,dy = -A\Big(\rho_1 c_{01}\int\limits_{-l_1}^{0} u_1 T_1\,dy + \rho_2 c_{02}\int\limits_{0}^{l_2} u_2 T_2\,dy\Big), \tag{3.7}$$

where

$$E_2(t) = \frac{1}{2}\,\rho_1 c_{01} \int\limits_{-l_1}^{0} T_1^2\,dy + \frac{1}{2}\,\rho_2 c_{02} \int\limits_{0}^{l_2} T_2^2\,dy. \tag{3.8}$$

Estimate (2.13) leads to

$$\int\limits_{-l_1}^{0} u_1^2\,dy \le \frac{2\delta_1^2 C_1\,\mathrm{e}^{-4\delta t}}{\rho_1}, \qquad \int\limits_{0}^{l_2} u_2^2\,dy \le \frac{2\delta_1^2 C_1\,\mathrm{e}^{-4\delta t}}{\rho_2}. \tag{3.9}$$

The functions $T_j(y,t)$ satisfy the Friedrichs inequalities (2.9); therefore from (3.7) we obtain an inequality similar to (2.10):

$$\frac{dE_2}{dt} + 4\delta_2 E_2(t) \le 2\delta_3 \sqrt{E_2(t)}\;\mathrm{e}^{-2\delta t},$$

where $\delta_2 = \min\,(l_1^{-2}\chi_1, l_2^{-2}\chi_2)$ and $\delta_3 = \sqrt{2}\,|A|\delta_1\sqrt{C_1}\,\max\,(\sqrt{c_{01}}, \sqrt{c_{02}})$. From this it follows that

$$E_2(t) \le \begin{cases} \delta_3^2(\mathrm{e}^{-2\delta t} - \mathrm{e}^{-2\delta_2 t})^2/[4(\delta_2-\delta)^2], & \delta_2 \ne \delta, \\ \delta_3^2 t^2\,\mathrm{e}^{-4\delta_2 t}, & \delta_2 = \delta. \end{cases} \tag{3.10}$$

Estimate (3.10) was derived taking into account that $E_2(0) = 0$ by virtue of (3.8) and initial data (3.6).

Estimates of the integrals

$$\int\limits_{-l_1}^{0} T_{1y}^2\,dy, \qquad \int\limits_{0}^{l_2} T_{2y}^2\,dy$$

are obtained from identity (2.14), in which ν_j needs to be replaced by χ_j, u_j by T_j, and F_j by $-Au_j$. Multiplying (2.14) by $\rho_j c_{0j}$ and combining the equalities obtained, we have the identity

$$\rho_1 c_{01} \int\limits_0^t \int\limits_{-l_1}^{0} (T_{1t}^2 + \chi_1^2 T_{1yy}^2)\,dy\,dt + \rho_2 c_{02} \int\limits_0^t \int\limits_0^{l_2} (T_{2t}^2 + \chi_2^2 T_{2yy}^2)\,dy\,dt$$

$$+\, k_1 \int\limits_{-l_1}^{0} T_{1y}^2\,dy + k_2 \int\limits_0^{l_2} T_{2y}^2\,dy = A^2\Big(\rho_1 c_{01} \int\limits_0^t \int\limits_{-l_1}^{0} u_1^2\,dy\,dt + \rho_2 c_{02} \int\limits_0^t \int\limits_0^{l_2} u_2^2\,dy\,dt\Big). \tag{3.11}$$

Using inequalities (3.9), from (3.11), we obtain

$$\int\limits_{-l_1}^{0} T_{1y}^2\,dy \le \frac{\delta_4(1-\mathrm{e}^{-4\delta t})}{k_1}, \qquad \int\limits_0^{l_2} T_{2y}^2\,dy \le \frac{\delta_4^2(1-\mathrm{e}^{-4\delta t})}{k_2}, \tag{3.12}$$

where

$$\delta_4 = A^2\delta_1^2 C_1(c_{01}+c_{02})/(2\delta).$$

Therefore, from (3.8)–(3.10) and (3.12), we obtain

$$|T_j(y,t)| \le \Big(2\sqrt{2\delta_4 E_2(t)/(k_j\rho_j c_{0j})}\;\Big)^{1/2}.$$

Hence, in this case, the temperature perturbations decay exponentially with time (as $\mathrm{e}^{-2\delta t}$ for $\delta \le \delta_2$ and as $\mathrm{e}^{-2\delta_2 t}$ for $\delta > \delta_2$).

Applying the Laplace transform to problem (3.1)–(3.6), we obtain the following boundary-value problem for the images:

$$\begin{aligned} \tilde{T}_1'' - \frac{p}{\chi_1}\,\tilde{T}_1 &= \frac{A\tilde{u}_1(y,p)}{\chi_1}, \qquad -l_1 < y < 0, \\ \tilde{T}_2'' - \frac{p}{\chi_2}\,\tilde{T}_2 &= \frac{A\tilde{u}_2(y,p)}{\chi_2}, \qquad 0 < y < l_2; \end{aligned} \tag{3.13}$$

$$\tilde{T}_1(0,p) = \tilde{T}_2(0,p), \qquad k\tilde{T}_1'(0,p) = \tilde{T}_2'(0,p); \tag{3.14}$$

$$\tilde{T}_1(-l_1,p) = 0, \qquad \tilde{T}_2(l_2,p) = 0 \tag{3.15}$$

($k = k_1/k_2$; the prime denotes differentiation with respect to y). The solution of problem (3.13) can be represented as

$$\tilde{T}_1(y,p) = L_1 \sinh\sqrt{\frac{p}{\chi_1}}\,y + L_2\cosh\sqrt{\frac{p}{\chi_1}}\,y + \frac{A}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{y} \tilde{u}_1(z,p)\sinh\sqrt{\frac{p}{\chi_1}}\,(y-z)\,dz; \tag{3.16}$$

$$\tilde{T}_2(y,p) = L_3 \sinh\sqrt{\frac{p}{\chi_2}}\,y + L_4\cosh\sqrt{\frac{p}{\chi_2}}\,y + \frac{A}{\chi_2\sqrt{p\chi_2^{-1}}}\int\limits_{0}^{y} \tilde{u}_2(z,p)\sinh\sqrt{\frac{p}{\chi_2}}\,(y-z)\,dz; \tag{3.17}$$

in this case, the functions $L_i(p)$ $(i=\overline{1,4})$ are found from (3.14) and (3.15):

$$\begin{gathered} L_1 = \frac{G_1(p)-G_2(p)}{W_2(p)}, \qquad L_2 = L_1\tanh\sqrt{\frac{p}{\chi_1}}\,l_1, \qquad L_3 = \frac{k}{\sqrt{\chi}}\,L_1 - G_1, \\ L_4 = L_2 - \frac{A}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\sinh\sqrt{\frac{p}{\chi_1}}\,z\,dz. \end{gathered} \tag{3.18}$$

Here

$$\begin{gathered} G_1(p) = -\frac{kA}{\chi_1}\sqrt{\frac{\chi_2}{p}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\cosh\sqrt{\frac{p}{\chi_1}}\,z\,dz, \\ G_2(p) = -\frac{A\coth\sqrt{p\chi_2^{-1}}\,l_2}{\chi_1\sqrt{p\chi_1^{-1}}}\int\limits_{-l_1}^{0}\tilde{u}_1(z,p)\sinh\sqrt{\frac{p}{\chi_1}}\,z\,dz + \frac{A}{\chi_2\sqrt{p\chi_2^{-1}}\,\sinh\sqrt{p\chi_2^{-1}}\,l_2}\int\limits_{0}^{l_2}\tilde{u}_2(z,p)\sinh\sqrt{\frac{p}{\chi_2}}\,(l_2-z)\,dz, \\ W_2(p) = \frac{k}{\sqrt{\chi}} + \tanh\sqrt{\frac{p}{\chi_1}}\,l_1\coth\sqrt{\frac{p}{\chi_2}}\,l_2. \end{gathered} \tag{3.19}$$

We find the steady-state solution of problem (3.1)–(3.5) [the initial data (3.6) are ignored in this case]. For the functions $T_1^0(y)$ and $T_2^0(y)$, we have the problem

$$\begin{gathered} T^0_{1yy} = \frac{A}{\chi_1}\,u_1^0(y), \qquad -l_1 < y < 0, \\ T^0_{2yy} = \frac{A}{\chi_2}\,u_2^0(y), \qquad 0 < y < l_2; \end{gathered} \tag{3.20}$$

$$\begin{gathered} T_1^0(-l_1) = 0, \qquad T_2^0(l_2) = 0, \\ T_1^0(0) = T_2^0(0), \qquad kT^0_{1y}(0) = T^0_{2y}(0), \qquad k = k_1/k_2. \end{gathered} \tag{3.21}$$

Substituting the functions $u_1^0(y)$ and $u_2^0(y)$ from (2.24) into the right sides of Eqs. (3.20), integrating, and performing simple transformations, from (3.20) and (3.21), we obtain

$$\begin{gathered} T_1^0(y) = \frac{Al_1^2 f_0}{2\chi_1\nu_1}\Big(-\frac{y^4}{12l_1^2} + \frac{(\mu-l^2)y^3}{6l_1 l(\mu+l)} + \frac{\mu(l+1)y^2}{2l(\mu+l)}\Big) + a_1 y + a_2, \\ T_2^0(y) = \frac{Al_2^2 f_0\mu}{2\chi_2\nu_1}\Big(-\frac{y^4}{12l_2^2} + \frac{(\mu-l^2)y^3}{6l_2(\mu+l)} + \frac{l(l+1)y^2}{2(\mu+l)}\Big) + ka_1 y + a_2, \end{gathered} \tag{3.22}$$

where the constants a_1 and a_2 are determined from the formulas

$$a_1 = \frac{Al_1^3 f_0}{24\chi_1\nu_1(\mu+l)(k+l)}\Big[l^3(5\mu l + 4\mu + l^2) - \chi\mu(\mu + 4l^2 + 5l)\Big],$$

$$a_2 = -\frac{Al_1 l_2^3 f_0}{24\chi_1\nu_1(\mu+l)(k+l)}\Big[kl^2(5\mu l + 4\mu + l^2) + \chi\mu(\mu + 4l^2 + 5l)\Big].$$

It can be proved that $\lim\limits_{t\to\infty} T_j(y,t) = T_j^0(y)$, i.e., with time, the temperature perturbations in the layers become steady-state if $\lim\limits_{t\to\infty} f(t) = f_0$. For this, it is sufficient to calculate the limits $\lim\limits_{p\to 0} p\tilde{T}_j(y,p)$. As an example, for $j = 1$ we transform expression (3.16) using (3.18):

$$\tilde{T}_1(y,p) = \frac{G_1(p) - G_2(p)}{W_2(p)\cosh\sqrt{p\chi_1^{-1}}\, l_1}\sinh\sqrt{\frac{p}{\chi_1}}\,(y+l_1) + \frac{A}{\chi_1\sqrt{\chi_1^{-1}p}}\int\limits_{-l_1}^{y}\tilde{u}_1(z,p)\sinh\sqrt{\frac{p}{\chi_1}}\,(y-z)\,dz. \qquad (3.23)$$

Next, we can substitute $\tilde{u}_j(y,p)$ from (2.21) into (3.18), (3.19), and (3.23) and obtain an explicit expression for $\tilde{T}_1(y,p)$. However, this expression is very cumbersome and is not given in the present paper. There is a simpler method for calculating the limit $\lim\limits_{p\to 0} p\tilde{T}_1(y,p)$ based on (3.23) and the obtained limits $\lim\limits_{p\to 0} p\tilde{u}_j(y,p) = u_j^0(y)$ using formulas (2.24). For $p \to 0$ ($\sinh x \approx x$, $\cosh x \approx 1$, and $x \to 0$), relation (3.19) leads to

$$W_2(p) \sim \frac{k+l}{\sqrt{\chi}}, \qquad pG_1(p) \sim -\frac{kA}{\chi_1\sqrt{\chi_2^{-1}p}}\int\limits_{-l_1}^{0} u_1^0(z)\,dz,$$

$$pG_2(p) \sim \frac{A}{\chi_1 l_2\sqrt{p\chi_2^{-1}}}\Big(-\int\limits_{-l_1}^{0} u_1^0(z)z\,dz + \chi\int\limits_{0}^{l_2} u_2^0(z)(l_2 - z)\,dz\Big).$$

The integrals on the right sides of these expressions are easily calculated with the use of formulas (2.24):

$$\int\limits_{-l_1}^{0} u_1^0(z)\,dz = \frac{f_0 l_1^3}{12\nu_1 l(\mu+l)}\,(4\mu l + 3\mu + l^2),$$

$$\int\limits_{-l_1}^{0} u_1^0(z)z\,dz = -\frac{f_0 l_1^4}{24\nu_1 l(\mu+l)}\,(3\mu l + 2\mu + l^2),$$

$$\int\limits_{0}^{l_2} u_2^0(z)\,dz = \frac{f_0 l_2^3\mu}{12\nu_1(\mu+l)}\,(\mu + 3l^2 + 4l);$$

therefore,

$$\lim_{p\to 0}\frac{pG_1(p) - pG_2(p)}{W_2(p)\cosh\sqrt{p\chi_1^{-1}}\, l_1}\sinh\sqrt{\frac{p}{\chi_1}}\,(y+l_1)$$

$$= -\frac{Af_0 l_2^3[kl^2(8\mu l + 6\mu + 2l^2) + l^3(3\mu l + 2\mu + l^2) - \mu\chi(\mu + 4l^2 + 5l)](y+l_1)}{24\nu_1\chi_1(\mu+l)(k+l)}. \qquad (3.24)$$

For $p \to 0$, the second term in (3.23) multiplied by p has the limit

$$\frac{A}{\chi_1}\int\limits_{-l_1}^{y} u_1^0(z)(y-z)\,dz = \frac{Af_0 l_1^2}{2\nu_1\chi_1}\Big(-\frac{y^4}{12l_1^2} + \frac{(\mu - l^2)y^3}{6l_1 l(\mu+l)} + \frac{\mu(l+1)y^2}{2l(\mu+l)} + \frac{l_1(8\mu l + 6\mu + 2l^2)y + l_1^2(3\mu l + 2\mu + l^2)}{12l(\mu+l)}\Big). \qquad (3.25)$$

Combining (3.24) and (3.25), we obtain the same formula (3.22) for $T_1^0(y)$. Similarly, it can be shown that $\lim\limits_{p\to 0} p\tilde{T}_2(y,p) = T_2^0(y)$.

For semibounded layers for $l_1, l_2 \to \infty$, from (3.16)–(3.19), we obtain ($\mathrm{Pr}_j = \nu_j/\chi_j \neq 1$) equations

$$\tilde{Z}_1(y,p) = \frac{A\tilde{f}(p)}{p^2}\Big[C_1 \exp\Big(\sqrt{\chi_1^{-1}p}\,y\Big) - \frac{\sqrt{\nu}\,(\rho-1)\mathrm{Pr}_1}{(\mu+\sqrt{\nu})(\mathrm{Pr}_1-1)}\exp\Big(\sqrt{\nu_1^{-1}p}\,y\Big) - 1\Big],$$

$$\tilde{Z}_2(y,p) = \frac{A\tilde{f}(p)}{p^2}\Big[C_2 \exp\Big(-\sqrt{\chi_2^{-1}p}\,y\Big) - \frac{\mu(\rho-1)\mathrm{Pr}_2}{(\mu+\sqrt{\nu})(1-\mathrm{Pr}_2)}\exp\Big(-\sqrt{\nu_2^{-1}p}\,y\Big) - \rho\Big], \tag{3.26}$$

where

$$\tilde{Z}_j(y,p) = \lim_{l_1,l_2\to\infty} \tilde{T}_j(y,p,l_1,l_2),$$

$$C_1 = \frac{\rho-1}{(\mu+\sqrt{\nu})(k+\sqrt{\chi})}\Big(\frac{\mu\sqrt{\chi}\,(1+\sqrt{\mathrm{Pr}_2})}{\mathrm{Pr}_2-1} + \frac{k(\sqrt{\chi}+\sqrt{\nu\mathrm{Pr}_1})}{\mathrm{Pr}_1-1}\Big), \tag{3.27}$$

$$C_2 = \frac{\rho-1}{(\mu+\sqrt{\nu})(k+\sqrt{\chi})}\Big(\frac{\mu(\sqrt{\chi\mathrm{Pr}_2}-k)}{\mathrm{Pr}_2-1} + \frac{k\sqrt{\nu}\,(\sqrt{\mathrm{Pr}_1}-1)+\sqrt{\chi}\,(k-\sqrt{\nu})}{\mathrm{Pr}_1-1}\Big).$$

The functions $\tilde{Z}_1$ and $\tilde{Z}_2$ satisfy boundary conditions (3.14) and Eqs. (3.13), in whose right sides, $\tilde{u}_j$ needs to be replaced by $\tilde{U}_j$ using relations (3.25).

From (3.26), applying the inverse Laplace transform (2.23), we obtain the following representations of the temperature perturbations:

$$\begin{aligned} Z_1(y,t) = A\int_0^t (t-\tau)f(\tau)\Big\{C_1\Big[\Big(1+\frac{y^2}{2\chi_1(t-\tau)}\Big)\mathrm{Erf}\Big(-\frac{y}{2\sqrt{\chi_1(t-\tau)}}\Big) \\ + \frac{1}{\sqrt{\pi}}\frac{y}{\sqrt{\chi_1(t-\tau)}}\exp\Big(-\frac{y^2}{4\chi_1(t-\tau)}\Big)\Big] \\ - \frac{\sqrt{\nu}\,(\rho-1)\mathrm{Pr}_1}{(\mu+\sqrt{\nu})(\mathrm{Pr}_1-1)}\Big[\Big(1+\frac{y^2}{2\nu_1(t-\tau)}\Big)\mathrm{Erf}\Big(-\frac{y}{2\sqrt{\nu_1(t-\tau)}}\Big) \\ + \frac{1}{\sqrt{\pi}}\frac{y}{\sqrt{\nu_1(t-\tau)}}\exp\Big(-\frac{y^2}{4\nu_1(t-\tau)}\Big)\Big] - 1\Big\}\,d\tau \end{aligned} \tag{3.28}$$

for $y < 0$ and

$$\begin{aligned} Z_2(y,t) = A\int_0^t (t-\tau)f(\tau)\Big\{C_2\Big[\Big(1+\frac{y^2}{2\chi_2(t-\tau)}\Big)\mathrm{Erf}\Big(\frac{y}{2\sqrt{\chi_2(t-\tau)}}\Big) - \frac{1}{\sqrt{\pi}}\frac{y}{\sqrt{\chi_2(t-\tau)}}\exp\Big(-\frac{y^2}{4\chi_2(t-\tau)}\Big)\Big] \\ - \frac{\mu(\rho-1)\mathrm{Pr}_2}{(\mu+\sqrt{\nu})(1-\mathrm{Pr}_2)}\Big[\Big(1+\frac{y^2}{2\nu_2(t-\tau)}\Big)\mathrm{Erf}\Big(\frac{y}{2\sqrt{\nu_2(t-\tau)}}\Big) \\ - \frac{1}{\sqrt{\pi}}\frac{y}{\sqrt{\nu_2(t-\tau)}}\exp\Big(-\frac{y^2}{4\nu_2(t-\tau)}\Big)\Big] - \rho\Big\}\,d\tau \end{aligned} \tag{3.29}$$

for $y > 0$ with the constants C_1 and C_2 from formulas (3.27).

If the Prandtl numbers are equal to unity ($\Pr_j = 1$), the solution has a different form:

$$Z_1(y,t) = A\int_0^t (t-\tau)f(\tau)\Big\{q_1\Big[\Big(1+\frac{y^2}{2\chi_1(t-\tau)}\Big)\operatorname{Erf}\Big(-\frac{y}{2\sqrt{\chi_1(t-\tau)}}\Big)+\frac{1}{\sqrt{\pi}}\,\frac{y}{\sqrt{\chi_1(t-\tau)}}\,\exp\Big(-\frac{y^2}{4\chi_1(t-\tau)}\Big)\Big]$$

$$+\frac{\sqrt{\chi}\,(\rho-1)}{2(\mu+\sqrt{\chi})}\,\frac{y}{\sqrt{\chi_1(t-\tau)}}\Big[\frac{2}{\sqrt{\pi}}\exp\Big(-\frac{y^2}{4\chi_1(t-\tau)}\Big)+\frac{y}{\sqrt{\chi_1(t-\tau)}}\operatorname{Erf}\Big(-\frac{y}{2\sqrt{\chi_1(t-\tau)}}\Big)\Big]-1\Big\}\,d\tau,$$

$$Z_2(y,t) = A\int_0^t (t-\tau)f(\tau)\Big\{q_2\Big[\Big(1+\frac{y^2}{2\chi_2(t-\tau)}\Big)\operatorname{Erf}\Big(\frac{y}{2\sqrt{\chi_2(t-\tau)}}\Big) \qquad (3.30)$$

$$-\frac{1}{\sqrt{\pi}}\,\frac{y}{\sqrt{\chi_2(t-\tau)}}\,\exp\Big(-\frac{y^2}{4\chi_2(t-\tau)}\Big)\Big]$$

$$+\frac{\mu(\rho-1)}{2(\mu+\sqrt{\chi})}\,\frac{y}{\sqrt{\chi_2(t-\tau)}}\Big[\frac{2}{\sqrt{\pi}}\exp\Big(-\frac{y^2}{4\chi_2(t-\tau)}\Big)-\frac{y}{\sqrt{\chi_2(t-\tau)}}\operatorname{Erf}\Big(\frac{y}{2\sqrt{\chi_2(t-\tau)}}\Big)\Big]-\rho\Big\}\,d\tau.$$

Here

$$q_1 = -\frac{\sqrt{\chi}\,(\rho-1)(k+\mu+2\sqrt{\chi}\,)}{2(\mu+\sqrt{\chi}\,)(k+\sqrt{\chi}\,)}, \qquad q_2 = \frac{(\rho-1)(2\mu k+\mu\sqrt{\chi}\,+k\sqrt{\chi}\,)}{4(\mu+\sqrt{\chi}\,)(k+\sqrt{\chi}\,)}.$$

For $f(t) = f_1/\sqrt{t}$ ($f_1 = \text{const}$), among solutions (2.26) and (3.28)–(3.30) there are self-similar solutions $U_j = \sqrt{t}\,a_j(\xi_j)$ and $Z_j = \sqrt{t^3}\,b_j(\xi_j)$, where $\xi_j = y/\sqrt{\nu_j t}$ because only in this case are Eqs. (2.1), (2.3) and (3.1), (3.3) invariant with respect to the group of stretching transformations $\bar{u} = \gamma u$, $\bar{y} = \gamma y$, $\bar{t} = \gamma^2 t$, and $\bar{T} = \gamma^3 T$ with the parameter γ.

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